Problems from Pi Mu Epsilon Journal, due March 31, 2024 (soft deadline)

#1400: *Proposed by Brian Ha and Steven Miller, Williams College.* Assume f satisfies a modified mean value property: for any $z \in \mathbb{C}$ there is some radius $r = r(z)$ such that $f(z)$ is the average of $f(\zeta)$ as ζ runs over the boundary of the circle of radius $r(z)$. Explicitly: there is a circle $\gamma(z)$ of radius $r(z)$ centered at z such that

$$\frac{1}{2\pi r(z)} \oint_{\gamma(z)} f(\zeta) d\zeta \;=\; f(z).$$

Prove or disprove: f is holomorphic (i.e., complex differentiable).

#1401: *Proposed by Hongwei Chen (Christopher Newport University).* Let m be a positive integer. For $|q| < 1$, prove

$$\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{m}\right) q^{n(n+1)/2} = \cos(\pi/m) \prod_{n=1}^{\infty} (1 + 2\cos(2\pi/m)q^n + q^{2n})(1 - q^n).$$

Use this result to deduce the recent Monthly problem 12289:

$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = \prod_{n=1}^{\infty} (1 - q^{6n-1})(1 - q^{6n-5})(1 - q^n).$$

Motivation. One of the most beautiful identity involving two parameters is perhaps Jacobi's triple product identity, which is given by

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1 + zq^{2n-1})(1 + z^{-1}q^{2n-1})(1 - q^{2n}). \tag{1}$$

Because its symmetric structure, the simplest proofs of many important theorems follow from this identity. For example, replacing q by q^3 in (1), then setting $z = -q^{-1}$, we immediately obtain Euler's famous pentagonal number theorem

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} = \prod_{n=1}^{\infty} (1 - q^{3n-1})(1 - q^{3n-2})(1 - q^{3n}) = \prod_{n=1}^{\infty} (1 - q^n).$$

Let $p(n)$ be the number of partitions of n. A product representation of the generating function for $p(n)$ has the form:

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.$$

#1402: *Proposed by Steven J. Miller (Williams College) and Elliott Weinstein (Centers for Medicare & Medicaid Services).* You buy a quick pick-k lottery ticket, that is, for fixed $k \geq 1$, a lottery ticket with k numbers $m_1, m_2, \ldots, m_k$ on it generated randomly without replacement from the numbers $1, 2, \ldots, N$ for some N, but you have no other information as to what N is. What is your best guess for N, and thus what is your best guess for the probability your ticket wins the jackpot, i.e., that all k numbers on your ticket match the k numbers in the lottery drawing?

#1403: *Proposed by Geoffrey Caveney, Veritas Tutor:* https://veritastutor.com/index.html. Mathematician 1 states that a constant A in a certain equation must be the product of an algebraic number and the (natural) logarithm of another algebraic number. Mathematician 2 gives a concrete example of such an equation, in which the constant A has the value $\frac{\pi}{7}4$. Which of the following statements is true?

- The example provided by Mathematician 2 contradicts the statement by Mathematician 1.
- The example provided by Mathematician 2 is consistent with the statement by Mathematician 1.
- The statement by Mathematician 1 together with the example provided by Mathematician 2 imply that the numbers π and e (the base of the natural logarithm) are not algebraically independent.
- The statement by Mathematician 1 together with the example provided by Mathematician 2 imply that the number e^π is an algebraic number.

#1404: *Proposed by Leo Hong, University of North Carolina at Charlotte.* Define a *great number* as a 10 digit number where each digit from 0 to 9 inclusive is used once and only once. (1) Does there exist a great number G whose double is also great? (2) How many great numbers G are there whose double is also great?

#1405: *Proposed by Steven J. Miller (Williams College), Rajaram Venkataramani and Anand Mohanram.* Let $p, p+2$ be odd twin primes at least 5; for example 5 and 7, 71 and 73, or 71,733,689 and 71,733,691. Multiply the two primes, and sum the digits. If the sum is not a one digit number, sum the digits again, and keep doing this until a one digit number arises. For example, for our three pairs we get 5 and 7 yields 35, so the digit sum is 8, while 71 and 73 has a product of 5,183 whose digit sum is 17 whose digit sum is 8, and the last pair's product is 5,145,722,281,016,099 is 62 which then gives a digit sum of 8. Is this a coincidence or will we always end with an 8?

Problems from Crux MathemAttic, due April 15, 2024

MA256. There exist positive integers whose value is quadrupled by moving the rightmost decimal digit into the leftmost position. Find the smallest such number.

MA257. A square of area 1 is divided into three rectangles which are geometrically similar (i.e., they have the same ratio of long to short sides) but no two of which are congruent. Let A, B and C be the areas of the rectangles, ordered from largest to smallest. Prove that $(AC)^2 = B^5$.

MA258. The three following circles are tangent to each other: the first has centre $(0, 0)$ and radius 4, the second has centre $(3, 0)$ and radius 1, and the third has centre $(-1, 0)$ and radius 3. Find the radius of a fourth circle tangent to each of these 3 circles.

MA259. Consider the equation $7a + 12b = c$ where a, b and c are nonnegative integers. For many values of c, it is possible to find one or more pairs (a, b) satisfying the equation. Given $c = 26$, for example, $(a, b) = (2, 1)$ is the only solution.

a) If $c = 365$, find all possible solutions (a, b), where a and b are nonnegative integers.

b) There are some values of c for which no solutions exist. For example, there is no pair (a, b) such that $7a + 12b = 20$, so $c = 20$ is one such case. Find the largest integer value of c for which there are no nonnegative integer solutions.

MA260. The expression $n!$ denotes the product $1 \cdot 2 \cdot 3 \cdots n$ and is read as "n factorial". For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

a) The product $(2!)(3!)(4!)(5!)(6!)(7!)(8!)(9!)(10!)(11!)(12!)$ can be written in the form $M^2N!$, where M, N are positive integers. Find a suitable value of N and justify your answer.

b) Prove that, for every $n \geq 1$, $(2!)(3!)(4!) \cdots ((4n)!)$ can be written as the product of a square and a factorial.

Problems from Crux Olympiad Corner, due April 15, 2024

OC666. The squares of a 1×10 board are numbered 1 to 10 in order. Clarissa and Marissa start from square 1, jump 9 times to the other squares so that they visit each square once, and end up at square 10. Jumps forward and backward are allowed. Each jump of Clarissa was for the same distance as the corresponding jump for Marissa. Does this mean that they both visited squares in the same order?

OC667. In the convex quadrilateral $ABCD$, AB and CD are parallel. Moreover, $\angle DAC = \angle ABD$ and $\angle CAB = \angle DBC$. Is $ABCD$ necessarily a square?

OC668. Consider all 100-digit positive integers such that each digit is 2, 3, 4, 5, 6, 7. How many of these integers are divisible by 2^{100}?

OC669. Let $M_2(\mathbb{Z})$ be the set of 2×2 matrices with integer entries. Let $A \in M_2(\mathbb{Z})$ such that

$$A^2 + 5I = 0,$$

where $I \in M_2(\mathbb{Z})$ and $0 \in M_2(\mathbb{Z})$ denote the identity and null matrices, respectively. Prove that there exists an invertible matrix $C \in M_2(\mathbb{Z})$ with $C^{-1} \in M_2(\mathbb{Z})$ such that

$$CAC^{-1} = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix} \quad \text{or} \quad CAC^{-1} = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}.$$

OC670. Prove that the arithmetic sequence $5, 11, 17, 23, 29, \ldots$ contains infinitely many primes.

Problems from Crux Mathematicorum, due April 15, 2024

4911. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Given a triangle ABC with $\angle BAC = 60°$, let P denote one of the points where its circumcircle intersects the perpendicular bisector of AC, and T denote the foot of the perpendicular from P to the bisector of $\angle BAC$. Prove that PT is tangent to the nine-point circle of ΔABC at T.

4912. *Proposed by Michel Bataille.*

Let P be a point inside an equilateral triangle ABC with side a. Prove that PA, PB and PC are the sides of a triangle $\mathcal{T}$ and that $\mathcal{T}$ has an angle of $60°$ if and only one of its medians has length $\frac{a}{2}$.

4913. *Proposed by Albert Natian.*

Suppose the continuous function f satisfies the integral equation

$$\int_0^{xf(7)} f\left(\frac{tx^2}{f(7)}\right) dt = 3f(7)\, x^4.$$

Find $f(7)$.

4914. *Proposed by Ivan Hadinata.*

Let $\mathbb{R}_{\geq 0}$ be the set of all non-negative real numbers. Find all possible monotonically increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying

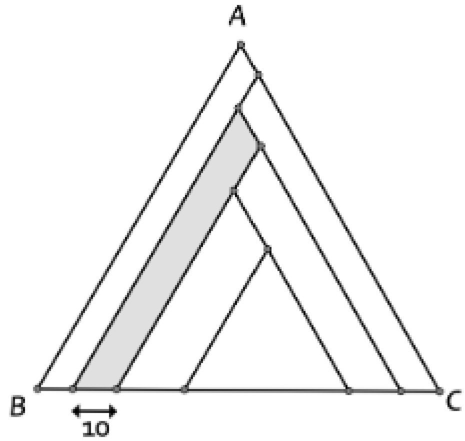
$$f(x^2 + y + 1) = xf(x) + f(y) + 1, \quad \forall x, y \in \mathbb{R}_{\geq 0}.$$

4915. *Proposed by Michel Bataille.*

Let $S_n = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+n+1)}$, where n is a nonnegative integer. Find real numbers a, b, c such that $\displaystyle\lim_{n\to\infty} \left(n^3 S_n - (an^2 + bn + c)\right) = 0$.

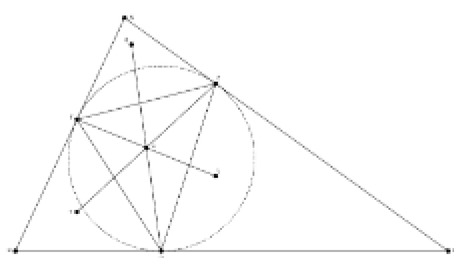
4916. *Proposed by Arsalan Wares.*

Equilateral triangle ABC is split into 5 isosceles trapezoids and a smaller equilateral triangle, all of the same area. One of the side lengths of the shaded trapezoid is 10. Determine the exact length of AB.



4917. *Proposed by Pericles Papadopoulos.*

Let D, E and F be the points of contact of the incircle of a triangle ABC with the sides BC, AC and AB, respectively. Let S, T and U be the orthocenters of the triangles EAF, FBD and DCE, respectively. Prove that SD, TE and UF concur at a point.



4918. *Proposed by Yagub Aliyev.*

Let $L = \lim_{\lambda \to +\infty} \dfrac{\lambda^{x^2}}{\int_a^b \lambda^{t^2} dt}$.

a) Show that if $0 \le a \le x < b$, then $L = 0$.

b) Show that if $0 \le a < x = b$, then $L = +\infty$.

4919. *Proposed by Daniel Sitaru.*

If $A, B \in M_6(\mathbb{R})$ are matrices such that
$$A^2 + B^2 = AB + A + B - I_6,$$
then
$$\det(BA - AB) \geq 0$$

4920. *Proposed by Ángel Plaza.*

If $k > 1$ and $n \in \mathbb{N}$, evaluate $\displaystyle\int_0^1 \frac{\log(1 + x^k + x^{2k} + \cdots + x^{nk})}{x} dx.$

Problems from the American Mathematical Monthly, due April 30, 2024

We reprint the following problem from the November 2023 issue, correcting an error.

12424. *Proposed by Proposed by Isaac Browne, Irvine, CA, and Edward Hou, Carnegie Mellon University, Pittsburgh, PA.* Prove that there is a sequence $L_1, L_2, \ldots$ of congruent convex sets in the plane such that for every finite set S of positive integers, the intersection

$$\bigcap_{i\in S} L_i \cap \bigcap_{i\notin S} L_i^c$$

has nonempty interior.

12426. *Proposed by Richard Kaufman, Office Expander, North Andover, MA, and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.*
(**a**) The $3n+1$ function takes n to $n/2$ if n is even and to $(3n+1)/2$ if n is odd. Show that for every positive integer m there exists a positive integer a such that am reaches 1 upon iteration of the $3n+1$ function.
(**b**) Show the same result for the $5n+1$ function, defined by replacing $3n+1$ by $5n+1$ in the definition in (**a**).

12427. *Proposed by Besfort Shala, University of Bristol, Bristol, UK.* Let $a_1, \ldots, a_n$ be real numbers such that $a_1 \geq 1$ and $a_{i+1} \geq a_i + 1$ for $1 \leq i \leq n-1$. Prove

$$\frac{\sum_{i=1}^n a_i^3}{\sum_{i=1}^n i^3} \geq \left(\frac{\prod_{i=1}^n a_i^i}{\prod_{i=1}^n i^i}\right)^{4/(n^2+n)}.$$

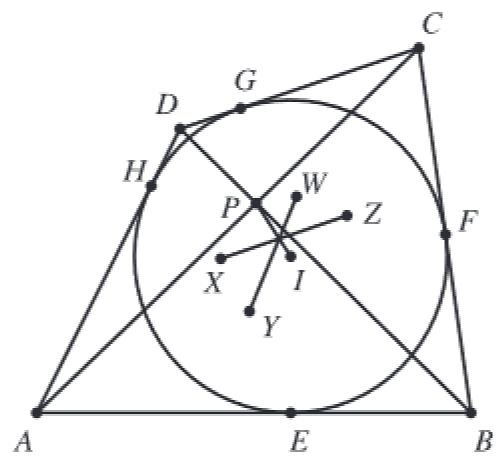
When does equality hold?

12428. *Proposed by William Weakley, Purdue University Fort Wayne, Fort Wayne, IN.* Let S be the set of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that, for every point $p \in \mathbb{R}^2$ except possibly the origin, the level set of f containing p is locally a curve that has a tangent line at p. For which angles α is there $f \in S$ such that rotating the family $\mathcal{F}$ of level curves of f clockwise about the origin by α gives the orthogonal family of $\mathcal{F}$?

12429. *Proposed by Leonard Giugiuc, Drobeta-Turnu Severin, Romania, and Mihai Prunescu, Bucharest, Romania.*
(**a**) For which values of c do there exist infinitely many pairs (a, b) such that $c \le \min\{a, b\}$ and $c = a + b - 2$ and such that there exists an acute triangle with sides of lengths a, b, and c whose altitude to the side of length c is less than c?
(**b**) Same question as (**a**) but with "acute" replaced by "obtuse".

12430. *Proposed by Askar Dzhumadil'daev, Almaty, Kazakhstan.* Let $P(n)$ denote the set of all partitions of $\{1, \ldots, n\}$. For $A = \{A_1, \ldots, A_k\}$ in $P(n)$, let $f(A) = \prod_{i=1}^{k} |A_i|$ and $g(A) = \prod_{i=1}^{k} \sum_{m \in A_i} m$.
(**a**) Prove $\sum_{A \in P(n)} f(A) = \sum_{k=1}^{n} k^{n-k} \binom{n}{k}$.
(**b**) Prove $\sum_{A \in P(n)} g(A) = \sum_{k=1}^{n} (n+1-k)^{k-1} c(n+1, k)$, where $c(n, k)$ is the unsigned Stirling number of the first kind, the number of permutations of $\{1, \ldots, n\}$ with exactly k cycles.

12431. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a *tangential* quadrilateral: This means that there is a circle tangent to all four sides. Let I be the center of that circle, and let P be the intersection of the diagonals of $ABCD$. Suppose that the circle touches the sides AB, BC, CD, and DA at points E, F, G, and H, respectively. Let W, X, Y, and Z be the centroids of triangles ECD, FDA, GAB, and HBC, respectively. Prove that the lines XZ, YW, and PI are concurrent.



12432. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* Suppose that k and n are integers with $n \ge 2$ and $1 \le k \le n$. What is the average value of $\sum_{i=1}^{\pi(k)} \pi(i)$ over all permutations π of $\{1, \ldots, n\}$?

Problems from Mathematics Magazine, due May 1, 2024

2181. *Proposed by Raymond Mortini, Université de Lorraine (emeritus), Metz, France, Peter Pflug, Carl von Ossietzky Universität Oldenburg (emeritus), Oldenburg, Germany, and Rudolf Rupp, Technische Hochschule Nürnberg Georg Simon Ohm, Nürnberg, Germany.*

Evaluate

$$\lim_{x\to\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k}{k!}\,\frac{(-1)^{m+1}}{(2(m+1))!}\,\frac{1}{2m+2+k}x^{2m+2+k}.$$

2182. *Proposed by Elton Bojaxhiu, Eppstein am Taunus, Germany, and Enkel Hysnelaj, Sydney, Australia.*

Let $n, m \geq 2$ be two natural numbers and let ℓ be a fixed nonzero integer. Find the number of functions

$$f : \mathbb{Z}_n \mapsto \mathbb{Z}_m,$$

such that for all $i \in \mathbb{Z}_n$,

$$f(i+\ell) \neq f(i).$$

2183. *Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Romania.*

If $A, B \in M_n(\mathbb{C})$ are two idempotent matrices, such that $\det(A-B) \neq 0$, show that

$$\operatorname{rank}(AB-BA) = \operatorname{rank}(AB+BA).$$

2184. *Proposed by the Columbus State University Problem Solving Group, Columbus State University, Columbus, GA.*

Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve

$$y = \frac{x}{x^2+1}$$

in exactly one point. (Be careful!)

2185. *Proposed by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

Suppose n is a nonnegative integer. Let $P_n(x)$ be the nth degree polynomial defined by

$$P_n(x) = \frac{(-1)^n(1+x^2)^{n+1}}{n!}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Evaluate

$$\int_{-1}^{1} P_n(x)\,dx.$$

Problems from the College Mathematics Journal, due May 15, 2024

1261. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.*

Evaluate the following integral for real $\alpha > -2$:

$$\int_0^\infty \frac{\tan^{-1}(x)}{1+\alpha x+x^2}dx.$$

1262. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let R and r be the circumradius and inradius, respectively, of triangle ABC. Choose D, E, and F on sides BC, CA, and AB, respectively, so that AD, BE, and CF bisect the angles of ABC. Prove that $(\frac{EF}{BC})^4 + (\frac{FD}{CA})^4 + (\frac{DE}{AB})^4 + \frac{3}{16} \leq \frac{3}{8}(\frac{R}{2r})^2$.

1263. *Proposed by Marius Munteanu, SUNY Oneonta, Oneonta, NY.*

Find, with proof, all solutions to the following equation, where $a \in [0, 1]$ is a fixed real number: $x(16x^2 - 20x + 5)^2 - a = 0$.

1264. *Stanescu Florin, Serban Cioculescu School, Gaesti, Romania.*

Let G be a finite group of order at least three, and $n \geq 2$ be an integer. Suppose that the function $f : G \to G$ defined by $f(x) := x^n$ is a surjective group homomorphism. Further, assume that for $2 \leq k \leq |G| - 1$, k does not divide $n - 1$. Prove that G is abelian.

1265. *Proposed by Narendra Bhandari, Bajura District, Nepal.*

Calculate the following sum:

$$\sum_{n=2}^{\infty}\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{\zeta(n)-\zeta(n+1)}{k},$$

where $[\frac{x}{\,}]$ denotes the floor function of x and ζ denotes the Riemann zeta function.

Problems from the American Mathematical Monthly, due May 31, 2024

12433. *Proposed by Etan Ossip, student, Queen's University, Kingston, ON, Canada.* For $x > 1$, prove

$$\frac{i}{2}\int_{-\infty}^{\infty} \frac{\tanh(\pi t)}{\left(\frac{1}{2}+it\right)^x}\, dt = \zeta(x),$$

where ζ is the Riemann zeta function.

12434. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Ploieşti, Romania.* Let $a_1, \dots, a_n$ be real numbers such that $a_1 \ge \cdots \ge a_n \ge 0$. Prove

$$\left(\frac{a_1a_2\cdots a_{n-1} + a_2a_3\cdots a_n + \cdots + a_na_1\cdots a_{n-2}}{n}\right)^2 \le \left(\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n}\right)^{n-1}.$$

12435. *Proposed by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy.* For a positive integer n, let $d(n)$ be the number of positive divisors of n, let $\phi(n)$ be Euler's totient function (the number of integers in $\{1, \dots, n\}$ that are relatively prime to n), and let $q(n) = d(\phi(n))/d(n)$. Find $\inf_n q(n)$ and $\sup_n q(n)$.

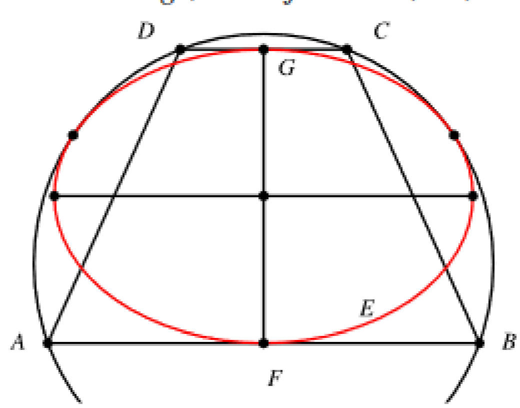
12436. *Proposed by Lorenzo Sauras-Altuzarra, Vienna University of Technology, Vienna, Austria.* For a positive integer n, evaluate

$$\prod_{k=1}^{n}\left(x + \sin^2\left(\frac{k\pi}{2n}\right)\right).$$

12437. *Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania, and Mihai Opincariu, Brad, Romania.* For any n-by-n complex matrix M, prove

$$\operatorname{rank}(M) + \operatorname{rank}\left(M - M^3\right) = \operatorname{rank}\left(M - M^2\right) + \operatorname{rank}\left(M + M^2\right).$$

12438. *Proposed by Robert Foote, Wabash College, Crawfordsville, IL, and Gregory Galperin, Eastern Illinois University, Charleston, IL.* Let $ABCD$ be an isosceles trapezoid inscribed in a circle γ with AB parallel to CD. Let F bisect AB and G bisect CD. Let E be the ellipse with minor axis FG and major axis of length AC. Prove that E is internally tangent to γ at two points.



12439. *Proposed by Haoran Chen, Xi'an Jiaotong–Liverpool University, Suzhou, China.* An n-by-n-by-n cube is formed from n^3 unit cubes. The removal of some of the unit cubes leaves a solid S such that

(1) the projection of S onto each face of the original cube is an n-by-n square; and
(2) from each unit cube in S one can reach any other unit cube in S along a chain of cubes each of which shares a face with its predecessor.

What is the minimum number of unit cubes that S can have?

Problems from Math Horizons, due May 31, 2024

A Curious Identity on Triangular Areas (P462). Yagub Aliyev (ADA University) posed this problem. As depicted in figure 1, segments AB and AC are two tangents from point A to some circle such that the triangle ABC is equilateral. Let X be an arbitrary point on the smaller arc $\widehat{BC}$. Suppose that the segments AX and BX intersect BC and AC at the points D and E, respectively. Prove that

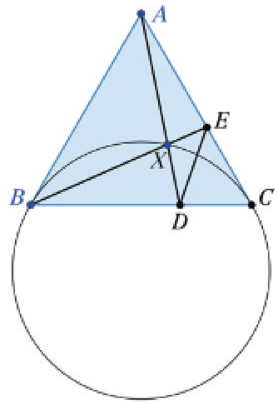


Figure 1. An equilateral triangle derived from a circle.

$$\sqrt{\frac{1}{[BDE]}+\frac{1}{[ADE]}}=\sqrt{\frac{1}{[ABD]}}+\sqrt{\frac{1}{[ACD]}}$$

where $[PQR]$ denotes the area of the triangle PQR.

Oppenheimer Numbers (P463). Brian J. Shelburne (Wittenberg University) asked this question. The book *American Prometheus*, which inspired the movie *Oppenheimer*, contains an intriguing story about J. Robert Oppenheimer and the number 3528. In short, Oppenheimer forgot the house number when searching for a New Year's party, only recalling interesting properties satisfied by the number. Fortunately, he eventually identified 3528 as the correct address.

Let $N = a_1 \ldots a_n b_1 \ldots b_n$ be a $2n$-digit number. Define N to be an *Oppenheimer number* if it satisfies the following conditions:

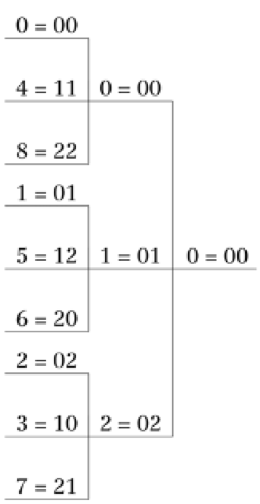
1) The two numbers created from the halves of N satisfy $\gcd(a_1 \ldots a_n, b_1 \ldots b_n) > 1$.
2) The digit a_1 is greater than 1 and divides N.
3) The digit b_n divides N.
4) The numbers a_1, $\gcd(a_1 \ldots a_n, b_1 \ldots b_n)$, and b_n are mutually coprime.

For example, 3528 is a four-digit Oppenheimer number. Is 3528 the only four-digit Oppenheimer number? Do $2n$-digit Oppenheimer numbers exist for all $n \geq 2$?

***Jeopardy*! Brackets (P464).** Stan Seltzer (Ithaca College) and Shai Simonson (Stonehill College) suggested this problem connected to their article “Bracket Challenge” (see page 20). The game show *Jeopardy*! features tournaments where each game consists of three contestants and one winner advancing. Define a seeded bracket with 3^k teams, using 0 to $3^k - 1$ as seeds, where three teams compete per game with one winner advancing to be *k-balanced* if the sum of the seeds is constant for each round when the top seed advances. For example, the nine-player bracket shown in figure 2 is 2-balanced.

A k-balanced bracket can be constructed using ternary (base 3) representations for the seeds as follows. Start with 0, 1, and 2 in the finals with 0 as the top seed. To generate the round with 3^2 teams, prepend a 0 to each (obtaining 00, 01, 02), and to each add 11 and 22 (base 3) with no carries. This gives the seeds expressed in ternary of the 2-balanced bracket in figure 2. Iterating produces a k-balanced bracket.

Figure 2. A 2-balanced bracket. The seeds are expressed using both decimal and (two-digit) ternary representations.



1) Using the 3-balanced bracket for 27 teams constructed as just described, what round would the 6 and 16 seeds meet?
2) What is the general algorithm to answer such a question for a k-balanced bracket constructed as just described?
3) There are other ways to construct k-balanced brackets. How many k-balanced brackets are there?

A Few Good Integer Polynomials (P465). Anthony Bevelacqua (University of Norh Dakota) proposed this problem. Let $p(x)$ be a polynomial with integer coefficients. Call $p(x)$ *good* if $p(x)$ takes the value 1 at exactly $r > 0$ distinct integers and $p(x)$ has exactly $s > 0$ distinct integer roots. For a good polynomial $p(x)$, show that (r, s) must be one of (1,1), (1,2), or (2,1).

Problems from Kappa Mu Epsilon's journal, *The Pentagon*, due May 31, 2024

Problem 920. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let α be the golden ratio. Show that

$$\sum_{i=0}^{\infty} \frac{i}{\alpha^i} \left(\sum_{j=0}^{\infty} \left[\lim_{n\to\infty} \frac{F_n^2+F_{n+2}^2-F_{n+1}F_{n+3}}{F_{n-1}F_{n+2}}\right]^j\right)^{-1} = \alpha$$

Problem 921. *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve in real numbers the following equation:

$$\log_2(x^2+2^x)+(x^2-1)*2^{x+1}+x^4+x^2+2^x=3*4^x+x+1$$

Problem 922. *Proposed by Toyesh Prakash Sharma,*

Let F_n be the nth Fibonacci number defined by $F_1 = 1, F_2 = 1$ and for all n≥3,$F_n = F_{n-1} + F_{n-2}$. Prove that $\sum_{n=1}^{\infty} (\frac{1}{9})^{F_{n+2}}$ is an irrational number but not a transcendental number.

Problem 923. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let n≥1 be an integer. Compute

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n*1}} \sum_{k=0}^{\infty} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

Problem 924. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let *a,b,c* be three positive real numbers. Prove that

$$\frac{a}{4b+7\sqrt{ab}} + \frac{b}{4c+7\sqrt{bc}} + \frac{c}{4a+7\sqrt{ca}} \geq \frac{3}{11}$$

Problem 925. *Proposed Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Prove that in any triangle ABC with usual notations (R = circumradius, r = inradius, s=semiperimeter, m_a = median from vertex A) the following inequality is true:

$$2\sum m_a \leq 3\sqrt{\frac{R(s^2+r^2+Rr)}{2r}}$$

Problem 926. *Proposed by the editor.*
Prove that the sequence
$a_1 = 1, a_2 = 1, a_n = a_{n-1} + 2 * a_{n-2}$ for all $n > 2$
gives the number of integers between 2^n and 2^{n+1} which are divisible by 3.

Problem 927. *Proposed by the editor.*

Find the area below the two lines 8x+5y= 976 and 6x+5y = 792 that lies in the first quadrant.